Measurement Error Analysis in Determination of Small-Body Gravity Fields

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Theme

ONE of the major objectives of future rendezvous-class missions to comets and asteroids is the determination of their internal structure. This may be derived from determination of total mass, mass distribution, and gravitation field. Such missions may carry onboard gravity gradiometers, TV cameras, and transponders for tracking. We are concerned with the use of such instruments and data types to achieve these science objectives. Simple potential-field models for a representative asteroid (Eros) and directly derived magnitudes of effects due to gravitational potential coefficients and to mass concentrations are given. Also presented is a covariance analysis based on Kalman filtering, to show science data accuracies achievable with representative measurements or data types, used singly and in combination.

Contents

Potential modeling: Small bodies must be regarded as having irregular figures. The body model selected is a constant-density triaxial ellipsoid with semi-axes a, b, c; in Cartesian coordinates, $(x/a)^2+(y/b)^2+(z/c)^2=1$. The potential V of such a body is given; in spherical coordinates r, L, λ,

$$V(r,L,\lambda)=\frac{GM}{r}\left[1+\sum_{n=2}\sum_{m=0}\left(\frac{Re}{r}\right)^n P_{nm}(\cos L)\cdot C_{nm}\cos m\lambda\right] \tag{1}$$

where $r^2=x^2+y^2+z^2$ (radial distance), $L=\sin^{-1}(z/r)$ (latitude), $\lambda=\tan^{-1}(y/x)$ (longitude), G=universal constant of gravitation, M=body mass, Re is a body characteristic dimension, P_{nm} is the associated Legendre polynomial, and the C_{nm} are numerical constants, the potential coefficients, found as functions of a, b, c. [1]

For Eros, the following parameters and resulting potential coefficients were adopted: $GM=4.794\times10^5$ m^3/sec^2, $a=17.5$ km, $b=8.0$ km, $c=3.5$ km; $C_{20}=-0.113$, $C_{22}=0.0396$, $C_{40}=0.0340$, $C_{42}=0.00319$, $C_{44}=0.000279$, with $C_{nm}=0$ for n or m odd. In Eq. (1), $Re=a$ was adopted. The gravity field components V_x ($\equiv\partial V/\partial x$), V_y, V_z were required, together with gravity gradient terms V_{xx} ($\equiv\partial^2V/\partial x^2$), V_{xy}, V_{xz}, V_{yy}, V_{yz}, V_{zz}. These were obtained by rewriting Eq. (1) in a new set of coordinates r, z, u_n, together with the auxiliary coordinate v_n;

$$r^2=x^2+y^2+z^2 \qquad u_n=\cos n\lambda\cos^nL$$
$$z=r\sin L \qquad v_n=\sin n\lambda\cos^nL$$

Note that $u_1=x/r$, $v_1=y/r$. The derivatives are:

$$\partial u_n/\partial x=(n/r)[u_{n-1}-(x/r)u_n],$$

$$\partial v_n/\partial x=(n/r)[v_{n-1}-(x/r)v_n]$$

$$\partial u_n/\partial y=(n/r)[-v_{n-1}-(y/r)u_n]$$

$$\partial v_n/\partial y=(n/r)[u_{n-1}-(y/r)v_n]$$

$$\partial u_n/\partial z=-(n/r^2)zu_n$$

$$\partial v_n/\partial z=-(n/r^2)zv_n$$

To study the detection of a mass concentration, the adopted potential model is that of two fixed point masses. Normalized units are used: the total gravitational mass GM is unity, the mascon has mass μ ($\mu<0.5$), the rest of the asteroid has mass $(1-\mu)$. The mascon is located at $x=1-\mu$, the rest of the asteroid at $x=-\mu$. Unit distance is the radius of the asteroid. Unit velocity $=(GM/r)^{1/2}$ at $r=1$. The potential is

$$V=(1-\mu)/r_1+\mu/r_2 \tag{2}$$

where $r_1{}^2=(x+\mu)^2+y^2+z^2$ (distance to larger mass point), $r_2{}^2=(x-1+\mu)^2+y^2+z^2$ (distance to mascon).

Orbits about Eros: To gain insight into the character of satellite orbits about small irregular bodies, we integrated several initially circular orbits about Eros. Variation with time of semimajor axis α and eccentricity e were followed by computing α, e from the integrated orbit state (x,y,z). The resulting variations were found to be large. An orbit initially with $e=0$, $\alpha=2a=35$ km, in the equatorial (xy) plane, had $31\leq\alpha\leq38$ km, $0\leq e\leq0.22$. Hence, close orbits appear out of the question since they will quickly impact the surface. Orbits with initial $\alpha\geq2a$ appear to be required.

Detection of triaxiality effects: Consider the detectability of the potential coefficients of Eros due to their separate effects, by means of the following data types: gravity gradiometry, velocity determination, and range determination. The effects to be detected are studied as a function of r along the x-axis. Let $\xi\equiv a=r$; then, for each of the C_{nm} and for GM, the

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associated contribution to the gravity gradient component V_{xx} is:

$$(V_{xx})_{00} \equiv \Gamma_{00} = 2GM/r^3 \qquad \Gamma_{40} = (45/8)\Gamma_{00}C_{40}\xi^4$$

$$\Gamma_{20} = 3\Gamma_{00}C_{20}\xi^2 \qquad \Gamma_{42} = (225/2)\Gamma_{00}C_{42}\xi^4$$

$$\Gamma_{22} = 18\Gamma_{00}C_{22}\xi^2 \qquad \Gamma_{44} = 1575\Gamma_{00}C_{44}\xi^4$$

Let the gravity gradiometry sensitivity limit be 3×10^{-10} gal/cm, achievable for 30 sec integration time [2] (1 gal = 1 cm/sec^2.) Then Γ_{00} is detectable to $r=9a$, Γ_{22} and Γ_{20} to $r=3a$, Γ_{40}, Γ_{42}, Γ_{44} to $r=2a$. The contributions to the gravity component V_x are:

$$(V_x)_{00} \equiv g_{00} = GM/r^2 \qquad g_{40} = (15/8)g_{00}C_{40}\xi^4$$

$$g_{20} = 6\, g_{00}C_{20}\xi^2 \qquad g_{42} = (75/2)g_{00}C_{42}\xi^4$$

$$g_{22} = 9\, g_{00}C_{22}\xi^2 \qquad g_{44} = 525\, g_{00}C_{44}\xi^4$$

The velocity perturbation ΔV_{nm} associated with C_{nm} was given by

$$\Delta V_{nm} = 2([(GM/r)(1+g_{nm})]^{1/2} - (GM/r)^{1/2}) \tag{3}$$

and the perturbation in range Δr_{nm} was given by

$$\Delta r_{nm} = (r^3/GM)^{1/2}\cdot\Delta V_{nm} \tag{4}$$

where $(r^3/GM)^{-1/2}$ is the orbital mean motion. By comparison with numerically integrated orbits, Eqs. (3) and (4) were found to be accurate to $\sim$10%. For a sensitivity limit in velocity determination of 10^{-3} m/sec, ΔV_{20} and ΔV_{22} are detectable beyond $r=10a$; ΔV_{40}, ΔV_{42}, ΔV_{44} are detectable to $r=4a$. For a range sensitivity of 100 m, however, Δr_{40}, Δr_{42}, Δr_{44} are detectable only to $r=2.5a$. Δr_{20}, Δr_{22} are detectable beyond $r=10a$.

Detection of mass concentrations: Initially circular orbits were integrated about a body with the potential of Eq. (2) and with $\mu=0.1$. The resulting perturbations in r, V, and Γ (gravity gradient) were nearly proportional in magnitude to μ. Of particular concern, in evaluating these data types, was the half-width of the peaks of the perturbation curves, or angular distance from the mascon at which the curve falls to half its maximum value. To convert from the normalized units of the integrations to physical units, values for GM and body dimension were taken as those of Eros. In particular, unit gravity gradient $= GM/a^3$.

For range as a data type, with sensitivity limit $\Delta r = 100$ m, at $r=2a$ one can detect $\mu \geq 0.005$; but the half-width is $\sim 90°$, indicating a very broad maximum. With velocity as a data type and sensitivity limit $\Delta V = 10^{-3}$ m/sec, at $r=2a$ one detects $\mu \geq 0.0005$ and the half-width is $\sim 60°$. Figure 1 shows traces of gravity gradient perturbation due to $\mu = 0.1$ as functions of mean anomaly, at $r=2a$. The half-width is now $<20°$ and with sensitivity limit 3×10^{-10} gal/cm, $\mu \geq 0.004$ is detectible. Hence, for mascon detection, range and velocity are not suitable as data types since they fail to give sharp peaks, indicating unambiguously a mascon's presence and location.

Statistical parameter estimation: Consider now the statistical estimation of gravitational parameters and orbit state. A standard form of the Kalman filter algorithm is assumed, which is equivalent to weighted least-squares processing of measurement data. The parameter estimation

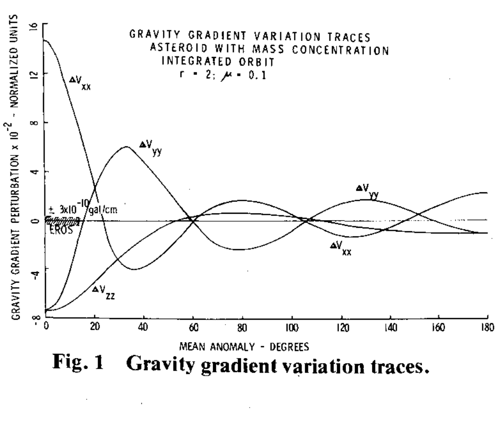

Fig. 1 Gravity gradient variation traces.

Table 1 RMS estimation errors after one orbit

Parameter	Units	DSN Doppler only (1)	Gravity Gradient only (2)	Ranging only (3)	Direction Angles only (4)	(1)+(3)+(4)	(2)+(3)+(4)
Position	m						
in-plane		62	22	304	26	2	6
out-of-plane		97	2588	2588	1	1	1
Velocity	10^{-3}						
in-plane	m/sec	7.1	2.0	50.8	3.8	0.3	0.6
out-of-plane		27.3	934.0	934.0	0.1	0.1	0.1
Mass	%	4.1	0.1	8.4	5.6	0.6	0.03
C_{20}	%	161.8	0.4	324.0	206.0	22.4	0.4
C_{22}	%	0.8	0.3	46.7	6.2	0.3	0.3
C_{40}	%	698.0	1.2	1359.0	850.6	94.4	1.2
C_{42}	%	11.2	2.0	532.0	65.4	3.5	1.7
C_{44}	%	0.7	0.6	91.8	8,2	0.5	0.3

study involves a covariance analysis only, and does not involve simulating observations and filtering them.

Four data types are considered, singly and in combination: 1) earth-based Doppler or range-rate; 2) onboard gravity gradiometry; 3) onboard ranging (e.g., with radar altimeter), and 4) onboard direction angles (via star tracking with a TV camera). Each measurement error is assumed statistically independent, unbiased (zero mean), time-uncorrelated, and with constant variance. The formulation for earth-based measurements is simplified by ignoring earth's rotation and station location errors and by fixing the asteroid's location.

A typical rendezvous mission to Eros arrives in March 1987, with Eros at distance 1.8 a.u. from earth and at declination 15.5°. The spacecraft is inserted into an initially circular orbit at 35 km from Eros' center, with period 14 hr. An equatorial orbit is assumed, the plane of which is inclined 66° to the earth-Eros line of sight. Initial 1σ or rms errors at orbit insertion are taken as 100 m in position, 0.5 m/sec in velocity. Initial uncertainty in knowledge of Eros' GM is 10%. The C_{nm} are considered unknown a priori. White-noise rms measurement errors are as follows: DSN Doppler, 10^{-3} m/sec; ranging, 100 m; gravity gradients, 3×10^{-10} gal/cm; direction angles, 20 arc sec. Measurements were assumed to be made at half-hour intervals. Doppler and gravity gradient measurements involve integration times, respectively, of 60 and 30 sec; their averaging-in at half-hour intervals reduces their effective rms noise values, respectively, to 1.8×10^{-4} m/sec and 3.9×10^{-11} gal/cm.

Results of the statistical error analysis are summarized in Table 1 which shows the rms errors obtained after one orbit of data processing. The effectiveness of each data type alone is given and may be compared with the results of combining several data types.

Onboard range measurements alone yield very poor estimates of all state variables and parameters. Direction angles alone do not give well-determined gravity coefficients but are quite effective in estimating position and velocity. Thus, this data type is most useful in combination with more direct information such as Doppler and gravity gradient measurements. Since DSN Doppler data provides direct velocity information only along the earth-spacecraft line of sight direction, its effectiveness is not uniform over the entire orbit and also depends on the orientation of the orbit plane. Gravity gradiometry yields the best estimates of the gravational field structure. All harmonic coefficients are accurate to with 2%. The addition of ranging and, particularly, direction angle data to gradiometry provides a very accurate estimate of spacecraft motion, but only yields a small improvement in harmonic coefficient determination over that of gradiometry alone. However, reference positional information would be necessary for any detailed mapping operation via gradiometry. An alternative mapping operation would combine DSN doppler with onboard ranging and direction angles. Multiple orbit data processing would probably be necessary to obtain an accurate estimate of the harmonic coefficients.

References

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[2] Forward, R. L., "Gravitational Field Measurements of Planetary Bodies with Gravity Gradient Instrumentation," AAS Paper 71-364, Ft. Lauderdale, Fla., Aug. 1971.